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# New results for the interior conductive eigenvalue problem

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## Abstract

In this paper, we consider problems associated with the interior eigenvalue problem for an inhomogeneous media with a conductive boundary. Transmission eigenvalue problems are a new class of eigenvalue problems that are not elliptic, not self-adjoint and nonlinear, which gives the possibility of complex eigenvalues. These eigenvalue problems are associated with the scattering by a penetrable obstacle. To begin, we first prove that there exists complex eigenvalues for a homogeneous circular/spherical obstacle with real valued coefficients. We then consider the problem for an absorbing media, proving discreteness for a general scatterer and existence of eigenvalues for the homogeneous circular/spherical. Next, we prove that the real interior conductive eigenvalues can be reconstructed for the scattering data using the linear sampling method. Lastly, we consider the limiting case where the conductivity parameter tends to zero and prove that the interior conductive eigenvalues and eigenfunctions converge linearly to the interior transmission eigenvalues and eigenfunctions, respectively. Several numerical results support the theory and we show that the inside-outside duality method can be used to reconstruct the interior conductive eigenvalues.

**Keywords:** inverse scattering, inhomogeneous medium, transmission eigenvalues, inverse spectral problem, conductive boundary condition.

## 1 Introduction

The development of qualitative (or direct) methods in the field of inverse acoustic and electromagnetic scattering has been a very active field of interest. The interior conductive eigenvalue problem can be derived by studying the injectivity of the operators used in qualitative inversion methods. Indeed, these values correspond to frequencies where the

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so called far field operator fails to be injective. This gives that there is an incident field (probing wave) that does not produce a scattered field in the exterior of the scattering object and therefore algorithms such as the linear sampling and factorization methods fail at those interior eigenvalues (see [3] and [17]). The interior conductive eigenvalues and their connection to inverse scattering was first introduced in [16]. The first proof of existence for non-radially symmetric scatterers was given in [23]. Recently, the interior transmission eigenvalue problem has become an important area of research, due to the fact that the interior transmission eigenvalues carry information about the material properties and can be determined from the scattering data (see for e.g. [5], [8], [18] and [27]). In [1] it is shown that the interior transmission eigenvalues depend monotonically on the material parameters for an inhomogeneous media with a conductive boundary. Therefore, the interior transmission eigenvalues can be used for non-destructive testing and target identification. The interior transmission eigenvalue problem is nonlinear and non self-adjoint which makes their investigation interesting from the mathematical point of view, but also a challenging task from the computational point of view.

The interior eigenvalue problem for an inhomogeneous media with a conductive boundary was studied in [1], where they used the theoretical results in [7] to prove existence of real eigenvalues. Their numerical experiments seem to indicate that the interior conductive eigenvalues can be reconstructed using the linear sampling method and that they converge to the interior transmission eigenvalues as the conductivity parameter goes to zero. The rest of the paper is structured as follows. We begin by proving the existence of complex-valued interior conductive eigenvalues for the homogeneous sphere and disk with real valued refractive index. Then we consider the discreteness (in a general scatterer) and existence (homogeneous sphere/disk) of the interior conductive eigenvalues for an absorbing media. Next, we show that the interior conductive eigenvalues can be recovered by plotting the regularized solution to the far field equation. Lastly, we show that as the conductivity parameter  $\eta \rightarrow 0$  the interior conductive eigenvalues and functions converge linearly to the eigenvalues and functions of the interior transmission eigenvalue problem. We present several numerical results showing the validity of the theoretical findings. Additionally, we show that the inside-outside duality method works as well.

## 2 Statement of the problem

The eigenvalue problem we consider here corresponds to the following direct scattering problem: find the total field  $u \in H_{loc}^1(\mathbb{R}^m)$  (where  $m = 2, 3$ ) such that

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^m \setminus \overline{D} \quad \text{and} \quad \Delta u + k^2 n u = 0 \quad \text{in } D \quad (1)$$

$$u^+ - u^- = 0 \quad \text{and} \quad \frac{\partial u^+}{\partial \nu} + \eta u^+ = \frac{\partial u^-}{\partial \nu} \quad \text{on } \partial D. \quad (2)$$

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The superscripts  $+$  and  $-$  indicate the trace on the boundary taken from the exterior or interior of the domain  $D$ , respectively. The total field is given by  $u = u^s + u^i$  where the incident field is given by  $u^i = e^{ikx \cdot d}$  with the incident direction  $d$  given by a point on the unit sphere/circle and the scattered field  $u^s$  satisfies the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r^{\frac{m-1}{2}} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0.$$

We let  $D \subset \mathbb{R}^m$  be a bounded simply connected open set with  $\nu$  being the unit outward normal to the boundary.

**Assumption 2.1.** *For physical and analytical considerations we assume that*

1. *the boundary  $\partial D$  is class  $\mathcal{C}^2$*
2.  *$n(x) \in L^\infty(D)$  is real-valued such that*

$$0 < n_{\min} = \inf_{x \in D} n(x) \quad \text{and} \quad \sup_{x \in D} n(x) = n_{\max} < \infty$$

3.  *$\eta(x) \in L^\infty(\partial D)$  is real-valued such that  $\eta > 0$ .*

With these assumptions we have that the forward problem is well-posed (see [2]) and the scattered field has the expansion (see for e.g. [3])

$$u^s(x, d) = \frac{e^{ik|x|}}{|x|^{\frac{m-1}{2}}} \left\{ u^\infty(\hat{x}, d) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \rightarrow \infty.$$

We now define the far field operator  $\mathcal{F} : L^2(\mathbb{S}) \mapsto L^2(\mathbb{S})$

$$(\mathcal{F}g)(\hat{x}) = \int_{\mathbb{S}} u_\infty(\hat{x}, d) g(d) \, ds(d) \quad \text{with } \mathbb{S} = \{\hat{x} \in \mathbb{R}^m : |\hat{x}| = 1\} \quad (3)$$

and the associated Herglotz wave function

$$v_g(x) = \int_{\mathbb{S}} e^{ikx \cdot d} g(d) \, ds(d) \quad \text{with } g(d) \in L^2(\mathbb{S}).$$

Notice that by superposition we have that  $\mathcal{F}g$  is the far field pattern to the scattered field corresponding to  $v_g$  replacing  $e^{ikx \cdot d}$  as the incident field. It can be shown (just as in Theorem 6.2 of [3]) that the far field operator  $\mathcal{F}$  is injective with a dense range provided that  $k$  is not an interior conductive eigenvalue. The eigenvalue problem with conductive boundary conditions reads: find  $k \in \mathbb{C}$  and nontrivial  $(w, v) \in L^2(D) \times L^2(D)$  where  $w - v \in H^2(D) \cap H_0^1(D)$  such that

$$\Delta w + k^2 n w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D \quad (4)$$

$$w - v = 0 \quad \text{and} \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \eta v \quad \text{on } \partial D \quad (5)$$

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where we define the Sobolev space

$$H^2(D) \cap H_0^1(D) = \{u \in H^2(D) : u = 0 \text{ on } \partial D\}$$

equipped with the  $H^2(D)$  norm.

Notice that from the conductive eigenvalue problem (4)–(5) we have that the difference  $u = w - v \in H^2(D) \cap H_0^1(D)$  satisfies

$$\Delta u + k^2 nu = -k^2(n-1)v \quad \text{in } D, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = \eta v \quad \text{on } \partial D.$$

This implies that the conductive eigenvalue problem can be written as: find the values  $k \in \mathbb{C}$  such that there is a nontrivial solution  $u \in H^2(D) \cap H_0^1(D)$  satisfying

$$(\Delta + k^2) \frac{1}{n-1} (\Delta u + k^2 nu) = 0 \quad \text{in } D, \tag{6}$$

$$-\frac{k^2}{\eta} \frac{\partial u}{\partial \nu} = \frac{1}{n-1} (\Delta u + k^2 nu) \quad \text{on } \partial D. \tag{7}$$

Notice that  $v$  and  $w$  are related to the eigenfunction  $u$  by

$$v = -\frac{1}{k^2(n-1)} (\Delta u + k^2 nu) \quad \text{and} \quad w = -\frac{1}{k^2(n-1)} (\Delta u + k^2 u).$$

It is clear that the conductive eigenvalue problem (4)–(5) is equivalent to the eigenvalue problem (6)–(7). To study the conductive eigenvalue problem we consider the variational problem associated with (6)–(7). Therefore, taking a test function  $\varphi \in H^2(D) \cap H_0^1(D)$ , multiplying (6) by  $\bar{\varphi}$  and integrate to obtain that

$$\begin{aligned} 0 &= \int_D \bar{\varphi} (\Delta + k^2) \frac{1}{n-1} (\Delta u + k^2 nu) \, dx \\ \iff 0 &= \int_D \bar{\varphi} \Delta \left( \frac{1}{n-1} (\Delta u + k^2 nu) \right) \, dx + k^2 \int_D \frac{1}{n-1} \bar{\varphi} (\Delta u + k^2 nu) \, dx. \end{aligned}$$

By Green's second theorem we have that

$$\begin{aligned} 0 &= \int_D \Delta \bar{\varphi} \left( \frac{1}{n-1} (\Delta u + k^2 nu) \right) \, dx + k^2 \int_D \frac{1}{n-1} \bar{\varphi} (\Delta u + k^2 nu) \, dx \\ &\quad - \int_{\partial D} \frac{\partial \bar{\varphi}}{\partial \nu} \left( \frac{1}{n-1} (\Delta u + k^2 nu) \right) \, ds \\ \iff 0 &= \int_D \frac{1}{n-1} (\Delta u + k^2 nu) (\Delta \bar{\varphi} + k^2 \bar{\varphi}) \, dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} \, ds \end{aligned}$$

where we have used the conductivity condition (7).

### 3 Complex-valued interior conductive eigenvalues

Due to the fact that conductive eigenvalue problems are non self-adjoint there is the possibility that complex-valued eigenvalues exist. For the case when  $\eta = 0$  one can compute complex-valued interior transmission eigenvalues numerically (see for e.g. [10] and [14]) but complex-valued eigenvalues have only been proven to exist for spherically stratified materials in [22]. The goal of this section is to prove the existence of complex-valued interior conductive eigenvalues for the case where  $\eta \neq 0$  with  $n \neq 1$  constant.

We begin this section by considering the conductive eigenvalue problem for a spherically media in  $\mathbb{R}^3$  with homogenous coefficients. Therefore, we assume that  $D = \{x \in \mathbb{R}^3 : |x| < 1\}$  where  $n$  and  $\eta$  are positive constants. We also assume that the eigenfunctions are radially-symmetric, therefore  $w(r)$  and  $v(r)$  with  $r = |x|$ . This implies that

$$r^2 w'' + 2rw' + k^2 r^2 n w = 0 \quad \text{and} \quad r^2 v'' + 2rv' + k^2 r^2 v = 0 \quad \text{in } D.$$

It is easy to see that  $w(r) = c_1 j_0(k\sqrt{n}r)$  and  $v(r) = c_2 j_0(kr)$  where  $j_0$  is the spherical Bessel function of the first kind of order zero with  $c_1$  and  $c_2$  constants. Applying the boundary conditions we have that  $k$  is a interior conductive eigenvalue if and only if

$$\det \begin{pmatrix} j_0(k\sqrt{n}) & -j_0(k) \\ k\sqrt{n}j_0'(k\sqrt{n}) & -kj_0'(k) - \eta j_0(k) \end{pmatrix} = 0.$$

Now recall that

$$j_0(t) = \frac{\sin t}{t} \quad \text{and} \quad j_0'(t) = \frac{-\sin t}{t^2} + \frac{\cos t}{t}$$

which gives that  $k$  is an interior conductive eigenvalue provided that  $k$  is a root of

$$d_\eta(k) = k \sin(k\sqrt{n}) \cos(k) - k\sqrt{n} \sin(k) \cos(k\sqrt{n}) + \eta \sin(k) \sin(k\sqrt{n}).$$

It has been proven in [22] that the function  $d_0(k)$  has infinitely many complex roots provided that  $\sqrt{n}$  is not an integer or a reciprocal of an integer. Now, notice that as  $\eta \rightarrow 0$  that  $d_\eta(k)$  converges uniformly to  $d_0(k)$  on every compact subset of the complex plane. Since,  $d_\eta(k)$  is an entire function we can conclude by Hurwitz's theorem (see p.g. 152 of [12]) that for  $\eta$  sufficiently small there exists infinitely many complex roots of  $d_\eta(k)$ .

**Theorem 3.1.** *Assume that  $n$  and  $\eta$  are constants where  $\sqrt{n}$  is not an integer or a reciprocal of an integer. Then there exists infinitely many complex-valued interior conductive eigenvalues for the sphere in  $\mathbb{R}^3$ , provided that  $\eta$  is sufficiently small.*

Now, consider the conductive eigenvalue problem for the unit disk in  $\mathbb{R}^2$ , where  $D = \{x \in \mathbb{R}^2 : |x| < 1\}$  with  $n \neq 1$  and  $\eta \neq 0$  positive constants. Just as in the previous case we assume that the eigenfunctions are radially-symmetric. Therefore,  $w(r)$  and  $v(r)$  with  $r = |x|$ , satisfy the differential equations

$$r^2 w'' + rw' + k^2 r^2 n w = 0 \quad \text{and} \quad r^2 v'' + rv' + k^2 r^2 v = 0 \quad \text{in } D.$$

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This implies that the eigenfunctions are given by  $w(r) = c_1 J_0(k\sqrt{n}r)$  and  $v(r) = c_2 J_0(kr)$  where  $J_0$  is the Bessel function of order zero with  $c_1$  and  $c_2$  constants. Applying the boundary conditions and using that  $J'_0 = -J_1$  gives that

$$\det \begin{pmatrix} J_0(k\sqrt{n}) & -J_0(k) \\ -k\sqrt{n}J_1(k\sqrt{n}) & kJ_1(k) - \eta J_0(k) \end{pmatrix} = 0$$

where  $J_1$  is the Bessel function of order 1. Therefore, we have that  $k$  is an interior conductive eigenvalue provided that  $d_\eta(k) = 0$  where

$$d_\eta(k) = kJ_0(k\sqrt{n})J_1(k) - k\sqrt{n}J_0(k)J_1(k\sqrt{n}) - \eta J_0(k\sqrt{n})J_0(k).$$

Just as the previous case the existence of infinitely many complex roots of  $d_0(k)$  was proven in [22]. Therefore, we can conclude the existence of infinitely many complex-valued interior conductive eigenvalues since  $d_\eta(k)$  is an analytic function and converges uniformly to  $d_0(k)$  on every compact subset of the complex plane.

**Theorem 3.2.** *Assume that  $n$  and  $\eta$  are positive constants. Then there exists infinitely many complex-valued interior conductive eigenvalues for the disk in  $\mathbb{R}^2$ , provided that  $\eta$  is sufficiently small.*

We have proven the existence of complex-valued interior conductive eigenvalues for the homogeneous sphere and disk. Another question one can ask is where in the complex plane can one expect to find the interior conductive eigenvalues. In [4] eigenvalue free zones in the complex plane are characterized for a homogenous scatterer with  $\eta = 0$ . One can also ask if the set of complex-valued interior conductive eigenvalues is a discrete set. In Theorem 3.1 of [1] the discreteness of the real eigenvalues is proven by connecting the interior conductive eigenvalues to the eigenvalues of a compact-matrix operator just as in [11]. Therefore, we can conclude that the set of complex-valued interior conductive eigenvalues is at most discrete since they correspond to the eigenvalues of a compact operator.

We now give an alternative proof of the discreteness for the set of interior conductive eigenvalues based on the Analytic Fredholm Theorem (see p.g. 13 of [3]). To this end, recall that the variational form of the conductive eigenvalue problem

$$0 = \int_D \frac{1}{n-1} (\Delta u + k^2 u) (\Delta \bar{\varphi} + k^2 n \bar{\varphi}) dx + k^2 \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds \quad (8)$$

for any  $\varphi \in H^2(D) \cap H_0^1(D)$ . Simple manipulation gives that the conductive eigenvalue problem can now be written in operator form as: find the values  $k \in \mathbb{C}$  such that there is a nontrivial solution  $u \in H^2(D) \cap H_0^1(D)$  satisfying

$$\mathbb{T}u + k^2 \mathbb{T}_1 u + k^4 \mathbb{T}_2 u = 0,$$

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where we define the bounded linear operators  $\mathbb{T}$ ,  $\mathbb{T}_1$  and  $\mathbb{T}_2 : H^2(D) \cap H_0^1(D) \mapsto H^2(D) \cap H_0^1(D)$  by the Riesz representation theorem such that

$$\begin{aligned} (\mathbb{T}u, \varphi)_{H^2(D)} &= \int_D \frac{1}{n-1} \Delta u \Delta \bar{\varphi} \, dx, \\ (\mathbb{T}_1 u, \varphi)_{H^2(D)} &= - \int_D \frac{1}{n-1} (\bar{\varphi} \Delta u + u \Delta \bar{\varphi}) \, dx + \int_D \nabla u \cdot \nabla \bar{\varphi} \, dx + \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} \, ds \end{aligned}$$

and

$$(\mathbb{T}_2 u, \varphi)_{H^2(D)} = \int_D \frac{n}{n-1} u \bar{\varphi} \, dx$$

for all  $\varphi \in H^2(D) \cap H_0^1(D)$ .

By appealing to the well-posedness of the Poisson's problem with zero trace on  $\partial D$  and elliptic regularity we have that in the space  $H^2(D) \cap H_0^1(D)$  the  $H^2(D)$  norm is equivalent to the  $L^2(D)$  norm of the Laplacian (see for e.g. [1]). It has been shown in [1] that the operators  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are self-adjoint and compact. We are now ready to prove the discreteness of the set of interior conductive eigenvalues.

**Theorem 3.3.** *Assume that  $n_{\min} > 1$  or  $0 < n_{\max} < 1$  then the set of interior conductive eigenvalues is at most discrete in  $\mathbb{C}$ .*

*Proof.* Since the  $H^2(D)$ -norm is equivalent to the  $L^2(D)$ -norm of the Laplacian in  $H^2(D) \cap H_0^1(D)$  the assumptions on  $n(x)$  gives that  $(\sigma \mathbb{T})^{-1}$  is a bounded linear operator on  $H^2(D) \cap H_0^1(D)$  by the Lax-Milgram lemma, where we define  $\sigma = 1$  when  $n_{\min} > 1$  and  $\sigma = -1$  when  $0 < n_{\max} < 1$ . Therefore, we have that if  $k \in \mathbb{C}$  is an interior conductive eigenvalue then

$$u + \sigma k^2 (\sigma \mathbb{T})^{-1} \mathbb{T}_1 u + \sigma k^4 (\sigma \mathbb{T})^{-1} \mathbb{T}_2 u = 0.$$

We now define the compact operator  $\mathbb{B}_k = \sigma k^2 (\sigma \mathbb{T})^{-1} \mathbb{T}_1 + \sigma k^4 (\sigma \mathbb{T})^{-1} \mathbb{T}_2$ . This implies that  $k$  is an interior conductive eigenvalue provided that the null space of  $\mathbb{I} + \mathbb{B}_k$  is non-trivial. By the definition of  $\mathbb{B}_k$  it is clear that the mapping  $k \mapsto \mathbb{B}_k$  is analytic for all  $k \in \mathbb{C}$ . Now notice that for  $k = 0$  that  $\mathbb{B}_0$  is the zero operator and we therefore have that  $\mathbb{I} + \mathbb{B}_0$  is injective. The result then follows directly from the Analytic Fredholm Theorem.  $\square$

We end this section by showing that for  $0 < n_{\max} < 1$  that there are no purely imaginary interior conductive eigenvalues.

**Theorem 3.4.** *Assume that  $0 < n_{\max} < 1$  then there are no purely imaginary interior conductive eigenvalues.*

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*Proof.* Indeed, assume  $k = i\tau$  for  $\tau \in \mathbb{R} \setminus \{0\}$  is an interior conductive eigenvalue, therefore by equation (25) in [1] we have that there is a  $u \neq 0$  such that

$$0 = \int_D \frac{n}{1-n} |\Delta u - \tau^2 u|^2 + |\Delta u|^2 dx + \tau^2 \|\nabla u\|_{L^2(D)}^2 + \int_{\partial D} \frac{\tau^2}{\eta} \left| \frac{\partial u}{\partial \nu} \right|^2 ds.$$

Since  $0 < n_{max} < 1$  we have that  $\|\Delta u\|_{L^2(D)}^2 = 0$ . Recall that the  $H^2(D)$ -norm is equivalent to the  $L^2(D)$ -norm of the Laplacian in  $H^2(D) \cap H_0^1(D)$ , which implies that  $u = 0$ , contradicting the fact that  $u$  is nontrivial.  $\square$

### 3.1 Conductive eigenvalues for absorbing media

In this section, we consider the case of an absorbing media. To this end, we reject the assumptions on the refractive index  $n$  given in assumption 2.1 and let the refractive index be given by

$$n(x) = n_1(x) + i \frac{n_2(x)}{k}$$

where  $n_\ell(x)$  are real-valued positive function in  $L^\infty(D)$  for  $\ell = 1, 2$ . The analysis of the transmission eigenvalue problem has already been studied in [6].

We start by considering the question of discreteness of the conductive eigenvalues for the case when  $n_2(x) \neq 0$ . Now, we rewrite the variational formulation (8) of the eigenvalue problem. Manipulating (8) we see that

$$\begin{aligned} 0 = & \int_D \frac{1}{n-1} \Delta u \Delta \bar{\varphi} dx - \int_D \frac{k^2}{n-1} (\bar{\varphi} \Delta u + u \Delta \bar{\varphi}) dx + k^2 \int_D \nabla u \cdot \nabla \bar{\varphi} dx \\ & + k^4 \int_D \frac{n}{n-1} u \bar{\varphi} dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds \end{aligned}$$

for all  $\varphi \in H^2(D) \cap H_0^1(D)$ . Substituting  $n = n_1 + i n_2/k$  into the variational formulation yields

$$\begin{aligned} 0 = & \int_D \frac{k}{k(n_1-1) + i n_2} \Delta u \Delta \bar{\varphi} dx + k^2 \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds + k^2 \int_D \nabla u \cdot \nabla \bar{\varphi} dx \\ & + k^4 \int_D \frac{k n_1 + i n_2}{k(n_1-1) + i n_2} u \bar{\varphi} dx - k^2 \int_D \frac{k}{k(n_1-1) + i n_2} (\bar{\varphi} \Delta u + u \Delta \bar{\varphi}) dx. \end{aligned}$$

Notice that we require that  $k(n_1-1) + i n_2 \neq 0$ , so we assume that  $n_1 - 1 \geq \gamma_1 > 0$  and  $n_2 \geq \gamma_2 > 0$  and that there is a

$$\delta > 0 \quad \text{such that} \quad \delta < n_2/(n_1-1)$$



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for all  $x \in D$ . This implies that  $|k(n_1 - 1) + in_2|$  is bounded below in the set

$$G_\delta = \{z \in \mathbb{C} \text{ such that } \operatorname{Im}(z) > -\delta\}.$$

The conductive eigenvalue problem can now be written in operator form as: find the values  $k \in G_\delta \subset \mathbb{C}$  such that there is a nontrivial solution  $u \in H^2(D) \cap H_0^1(D)$  satisfying

$$(\mathbb{A}_k + \mathbb{B}_k)u = 0$$

where the bounded linear operators  $\mathbb{A}_k$  and  $\mathbb{B}_k : H^2(D) \cap H_0^1(D) \mapsto H^2(D) \cap H_0^1(D)$  are defined by the Riesz representation theorem such that

$$\begin{aligned} (\mathbb{A}_k u, \varphi)_{H^2(D)} &= \int_D \frac{1}{k(n_1 - 1) + in_2} \Delta u \Delta \bar{\varphi} \, dx, \\ (\mathbb{B}_k u, \varphi)_{H^2(D)} &= k \int_{\partial D} \frac{1}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} \, ds + k \int_D \nabla u \cdot \nabla \bar{\varphi} \, dx + k^3 \int_D \frac{kn_1 + in_2}{k(n_1 - 1) + in_2} u \bar{\varphi} \, dx \\ &\quad - k^2 \int_D \frac{1}{k(n_1 - 1) + in_2} (\bar{\varphi} \Delta u + u \Delta \bar{\varphi}) \, dx \end{aligned}$$

for all  $\varphi \in H^2(D) \cap H_0^1(D)$ . Just as in the previous section we see that the operator  $\mathbb{B}_k$  is compact by appealing to Rellich's embedding theorem and the compact embedding of  $H^{1/2}(\partial D)$  into  $L^2(\partial D)$ .

To prove the discreteness we will use the Analytic Fredholm Theorem. To this end, notice that the mappings  $k \mapsto \mathbb{A}_k$  and  $\mathbb{B}_k$  are analytic for all  $k \in G_\delta$ . Next, we show that  $\mathbb{A}_k$  is coercive for all  $k \in G_\delta$ , therefore let  $k = a + ib$  which gives that

$$k(n_1 - 1) + in_2 = a(n_1 - 1) + i[n_2 + b(n_1 - 1)] := \alpha + i\beta.$$

This implies that the sesquilinear form for  $\mathbb{A}_k$  is given by

$$(\mathbb{A}_k u, \varphi)_{H^2(D)} = \int_D \frac{\alpha - i\beta}{|\alpha|^2 + |\beta|^2} \Delta u \Delta \bar{\varphi} \, dx.$$

Now, we obtain

$$-\operatorname{Im}(\mathbb{A}_k u, u)_{H^2(D)} = \int_D \frac{\beta}{|\alpha|^2 + |\beta|^2} |\Delta u|^2 \, dx$$

and since  $k \in G_\delta$  we can conclude that  $\beta = n_2 + b(n_1 - 1) > 0$ . Recall, that the  $L^2(D)$ -norm of the Laplacian is equivalent to the  $H^2(D)$ -norm in  $H^2(D) \cap H_0^1(D)$ , giving that  $\mathbb{A}_k$  is coercive for all  $k \in G_\delta$ . We now have all we need to prove the discreteness of the conductive eigenvalues in the set  $G_\delta \subset \mathbb{C}$ .

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**Theorem 3.5.** Assume that  $n_1 - 1 \geq \gamma_1 > 0$  and  $n_2 \geq \gamma_2 > 0$  and that

$$\delta > 0 \quad \text{is such that} \quad \delta < n_2/(n_1 - 1)$$

for all  $x \in D$ . Then the set of interior conductive eigenvalues is at most discrete in  $G_\delta$ .

*Proof.* We have that for all  $k \in G_\delta$  that  $\mathbb{A}_k^{-1}$  exists as a bounded linear operator from  $H^2(D) \cap H_0^1(D)$  to itself. Since  $\mathbb{A}_k$  depends analytically on  $k$  in  $G_\delta$  we can conclude that  $\mathbb{A}_k^{-1}$  is analytic. This gives that  $k$  is an interior conductive eigenvalue if and only if  $\mathbb{I} + \mathbb{A}_k^{-1}\mathbb{B}_k$  has a nontrivial kernel. Since the mapping  $k \mapsto \mathbb{A}_k^{-1}\mathbb{B}_k$  is analytic and compact we only need to prove that  $\mathbb{I} + \mathbb{A}_k^{-1}\mathbb{B}_k$  is injective at some point in  $G_\delta$  by the Analytic Fredholm Theorem. To this end, notice that  $\mathbb{B}_0$  is the zero operator and therefore  $\mathbb{I} + \mathbb{A}_0^{-1}\mathbb{B}_0$  is injective, proving the claim.  $\square$

Now, consider the case of an absorbing media with homogeneous refractive index. Therefore let the refractive index be given by  $n = n_1 + in_2/k$  where  $n_1$  and  $n_2$  are positive constants. As in the previous section we will consider the case of the unit circle and sphere. It is clear that the conductive eigenvalues corresponding to radially symmetric eigenfunctions are given by the roots of the function

$$\tilde{d}_\eta(k) = \begin{cases} kJ_0(k\tilde{n})J_1(k) - k\tilde{n}J_0(k)J_1(k\tilde{n}) - \eta J_0(k\tilde{n})J_0(k) & \text{in } \mathbb{R}^2 \\ k \sin(k\tilde{n}) \cos(k) - k\tilde{n} \sin(k) \cos(k\tilde{n}) + \eta \sin(k) \sin(k\tilde{n}) & \text{in } \mathbb{R}^3 \end{cases}$$

where  $\tilde{n} = \sqrt{n_1 + i\frac{n_2}{k}}$ . Now consider the limit as  $n_2 \rightarrow 0$ , then using a continuity argument we can conclude that  $\tilde{d}_\eta(k)$  converges to

$$d_\eta(k) = \begin{cases} kJ_0(k\sqrt{n_1})J_1(k) - k\sqrt{n_1}J_0(k)J_1(k\sqrt{n_1}) - \eta J_0(k\sqrt{n_1})J_0(k) & \text{in } \mathbb{R}^2 \\ k \sin(k\sqrt{n_1}) \cos(k) - k\sqrt{n_1} \sin(k) \cos(k\sqrt{n_1}) + \eta \sin(k) \sin(k\sqrt{n_1}) & \text{in } \mathbb{R}^3 \end{cases}$$

for all  $k \neq 0$  and the convergence is uniform on any compact subset of the complex plane not containing zero. Therefore, by appealing to Hurwitz's theorem we have the following result.

**Theorem 3.6.** Assume that  $n_1$ ,  $n_2$  and  $\eta$  are positive constants. Then there exists infinitely many interior conductive eigenvalues for the homogenous absorbing circle and sphere provided that  $n_2$  is sufficiently small.

## 4 Determination of interior conductive eigenvalues from scattering data

In this section, we show that the real interior conductive eigenvalues corresponding to our problem can be determined from the far field measurements following the approach in [5]

(also see [15]). We now introduce the far field equation, so let the radiating fundamental solution to Helmholtz equation in  $\mathbb{R}^m$  be denoted by

$$\Phi(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} & \text{in } \mathbb{R}^3 \end{cases}$$

and

$$\Phi_\infty(\hat{x}, z) = \gamma e^{ik\hat{x} \cdot z} \quad \text{with} \quad \gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \text{ for } m=2 \quad \text{and} \quad \gamma = \frac{1}{4\pi} \text{ for } m=3$$

is the far field pattern of  $\Phi(x, y)$  where  $H_0^{(1)}$  is the first kind Hankel function of order zero. We now define the far field equation for a fixed  $z \in D$  as:

$$\text{find } g_z \in L^2(\mathbb{S}) \quad \text{such that} \quad (\mathcal{F}g_z)(\hat{x}) = \Phi_\infty(\hat{x}, z) \quad (9)$$

where  $\mathcal{F}$  is the far field operator corresponding to the scattering problem (1)–(2).

Let  $\mathcal{F}^\delta$  be the far field operator corresponding to the “noisy” measurements  $u_\infty^\delta(\hat{x}, d)$  such that  $u_\infty^\delta(\hat{x}, d) \rightarrow u_\infty(\hat{x}, d)$  uniformly as  $\delta \rightarrow 0$ . Since the far field operator is compact we have that equation (9) is ill-posed, so we find the Tikhonov regularized solution  $g_{z,\delta}$  of the far field equation which is defined as the unique minimizer of

$$\|\mathcal{F}^\delta g - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S})}^2 + \epsilon \|g\|_{L^2(\mathbb{S})}^2$$

where the regularization parameter can be chosen by Morozov discrepancy principle such that  $\epsilon := \epsilon(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Now, assume that the regularized solution satisfies

$$\lim_{\delta \rightarrow 0} \|\mathcal{F}^\delta g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S})} = 0, \quad (10)$$

which holds provided that  $\mathcal{F}$  has a dense range. It can be shown that the range of  $\mathcal{F}$  is dense provided that  $k$  is not an interior conductive eigenvalue where the eigenfunction that satisfies the Helmholtz equation is a Herglotz wave function.

We now introduce the interior conductive problem given by: find the pair  $(w, v) \in H^1(D) \times H^1(D)$  such that

$$\Delta w + k^2 n w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D \quad (11)$$

$$w - v = \Phi(\cdot, z) \quad \text{and} \quad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} - \eta w = \frac{\partial}{\partial \nu} \Phi(\cdot, z) \quad \text{on } \partial D. \quad (12)$$

Now let  $u = v - w \in H^1(D)$  which implies that

$$\Delta u + k^2 u = k^2(n-1)w \quad \text{in } D \quad \text{and} \quad u = -\Phi(\cdot, z) \quad \text{on } \partial D.$$

By elliptic regularity we can conclude that  $u \in H^2(D)$  (see for e.g. [13]). Therefore, dividing by  $n - 1$  and applying  $(\Delta + k^2 n)$  gives that the interior conductive problem (11)–(12) can be written as

$$(\Delta + k^2 n) \frac{1}{n-1} (\Delta u + k^2 u) = 0 \quad \text{in } D \quad (13)$$

with the boundary conditions

$$u = -\Phi(\cdot, z) \quad \text{and} \quad \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} + \frac{1}{n-1} (\Delta u + k^2 u) = -\frac{k^2}{\eta} \frac{\partial}{\partial \nu} \Phi(\cdot, z) \quad \text{on } \partial D. \quad (14)$$

Now by taking a test function  $\varphi \in H^2(D) \cap H_0^1(D)$ , multiplying (13) by  $\bar{\varphi}$  and applying Green's second theorem gives that

$$\int_D \frac{1}{n-1} (\Delta u + k^2 u) (\Delta \bar{\varphi} + k^2 n \bar{\varphi}) dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds = - \int_{\partial D} \frac{k^2}{\eta} \frac{\partial \bar{\varphi}}{\partial \nu} \frac{\partial}{\partial \nu} \Phi(\cdot, z) ds$$

where we have used the conductive boundary condition in (14). We now take a lifting of the essential boundary data such that  $\phi_z \in H^2(D)$  and  $\phi_z = \Phi(\cdot, z)$  on  $\partial D$ . Now let  $u_0 \in H^2(D) \cap H_0^1(D)$  be given by  $u = u_0 - \phi_z$ . This implies that  $u_0$  is a solution to the variational problem

$$a(u_0, \varphi) = \ell(\varphi) \quad (15)$$

where the bounded sesquilinear form  $a(\cdot, \cdot)$  on  $H^2(D) \cap H_0^1(D) \times H^2(D) \cap H_0^1(D)$  is given by

$$a(u_0, \varphi) = \int_D \frac{1}{n-1} (\Delta u_0 + k^2 u_0) (\Delta \bar{\varphi} + k^2 n \bar{\varphi}) dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial u_0}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds$$

and the bounded conjugate linear functional  $\ell(\cdot)$  on  $H^2(D) \cap H_0^1(D)$  is given by

$$\begin{aligned} \ell(\varphi) = & \int_D \frac{1}{n-1} (\Delta \phi_z + k^2 \phi_z) (\Delta \bar{\varphi} + k^2 n \bar{\varphi}) dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial \phi_z}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} ds \\ & - \int_{\partial D} \frac{k^2}{\eta} \frac{\partial \bar{\varphi}}{\partial \nu} \frac{\partial}{\partial \nu} \Phi(\cdot, z) ds. \end{aligned}$$

The analysis in [1] shows that (15) satisfies the Fredholm property and since the sesquilinear form that defines the left hand side of (15) is Hermitian (see Theorem 4.1 of [1]) we have that if  $k$  is a real interior conductive eigenvalue with eigenfunction  $u_k$  then (15) has a solution if and only if  $\ell(u_k) = 0$  by the Fredholm Alternative.

**Theorem 4.1.** *Let  $k$  be a real interior conductive eigenvalue and assume that either  $n_{\min} > 1$  or  $0 < n_{\max} < 1$ . Then for a.e.  $z \in D$ ,  $\|g_{z,\delta}\|_{L^2(S)}$  can not be bounded as  $\delta \rightarrow 0$ , provided  $g_{z,\delta}$  satisfies (10).*

---

*Proof.* Assume that  $k$  is an interior conductive eigenvalue and that there is a set  $D_0 \subset D$  with  $|D_0| \neq 0$  where  $\|g_{z,\delta}\|_{L^2(\mathbb{S})}$  is bounded for all  $z \in D_0$ . Therefore, we can extract a subsequence such that the associated Herglotz wave function

$$v_{g_{z,\delta_n}} = \int_{\mathbb{S}} e^{ikx \cdot d} g_{z,\delta_n}(d) \, ds(d)$$

is weakly convergent to  $v \in H^1(D)$ , moreover  $\Delta v + k^2 v = 0$  in  $D$ . Notice that

$$\|\mathcal{F}g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S})} \leq \|\mathcal{F} - \mathcal{F}^\delta\| \|g_{z,\delta}\|_{L^2(\mathbb{S})} + \|\mathcal{F}^\delta g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S})}$$

which implies that

$$\|\mathcal{F}g_{z,\delta} - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S})} \longrightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.$$

Recall, that  $\mathcal{F}g_{z,\delta}$  is the far field pattern for the scattering problem (1)–(2) corresponding to  $v_{g_{z,\delta}}$  replacing  $e^{ikx \cdot d}$  as the incident field. Therefore, letting  $\delta \rightarrow 0$  and by appealing to Rellich's lemma we conclude that the scattered field  $u^s \in H_{loc}^1(\mathbb{R}^m)$  produced by the incident field  $v$  is given by  $\Phi(\cdot, z)$  on  $\mathbb{R}^m \setminus \bar{D}$ . This gives that the corresponding total field  $w = u^s + v$  in  $D$  and incident field  $v$  satisfy the interior problem (11)–(12), which gives that there is a solution to (15). Using Green's second theorem on the Fredholm solvability condition  $\ell(u_k) = 0$  along with

$$\phi_z = \Phi(\cdot, z) \quad \text{on} \quad \partial D \quad \text{and} \quad (\Delta + k^2) \frac{1}{n-1} (\Delta + k^2 n) u_k = 0 \quad \text{in} \quad D,$$

we obtain that

$$-\int_{\partial D} \frac{k^2}{\eta} \frac{\partial \bar{u}_k}{\partial \nu} \frac{\partial}{\partial \nu} \Phi(x, z) \, ds_x - \int_{\partial D} \Phi(x, z) \frac{\partial}{\partial \nu} \frac{1}{n-1} \overline{(\Delta u_k + k^2 n u_k)} \, ds_x = 0.$$

Now using the conductive boundary condition

$$-\frac{k^2}{\eta} \frac{\partial u_k}{\partial \nu} = \frac{1}{n-1} (\Delta u_k + k^2 n u_k)$$

we have that

$$\int_{\partial D} \frac{1}{n-1} \overline{(\Delta u_k + k^2 n u_k)} \frac{\partial}{\partial \nu} \Phi(x, z) - \Phi(x, z) \frac{\partial}{\partial \nu} \frac{1}{n-1} \overline{(\Delta u_k + k^2 n u_k)} \, ds_x = 0.$$

Now define  $\psi = \frac{1}{n-1} \overline{(\Delta u_k + k^2 n u_k)}$  which is a solution to the Helmholtz equation in  $D$  and is therefore analytic in the interior of  $D$ . Green's representation theorem gives that  $\psi(z) = 0$  for all  $z \in D_0$  and by unique continuation we conclude that  $\psi = 0$  in  $D$ . This implies that  $\Delta u_k + k^2 n u_k = 0$  in  $D$  and  $\partial u_k / \partial \nu = 0$  on  $\partial D$ . Now since  $u_k = \partial u_k / \partial \nu = 0$  on  $\partial D$  the unique continuation principle implies that  $u_k = 0$  in  $D$  which contradicts the fact that  $u_k$  is an eigenfunction.  $\square$

By Theorem 4.1 we have a way of computing the interior conductive eigenvalues *without a-priori* knowledge of the parameters  $n$  and  $\eta$ . In [1] it is shown that the first interior conductive eigenvalue can be reconstructed for a constant  $n$  or  $\eta$  provided one of the parameters is known. Therefore, one can obtain qualitative information about the material properties from the measured far field data.

**An explicit example for the unit ball in  $\mathbb{R}^3$ :** We will now give an explicit example where we assume that the  $D$  is the unit sphere. We will show that the solution to the far field equation (9) becomes unbounded as  $k$  approaches an interior conductive eigenvalue. We will now assume that the coefficients  $n$  and  $\eta$  are constant. Separation of variables gives that the interior conductive eigenvalues satisfy

$$\det \begin{pmatrix} j_p(k\sqrt{n}) & -j_p(k) \\ k\sqrt{n}j_p'(k\sqrt{n}) & -kj_p'(k) - \eta j_p(k) \end{pmatrix} = 0$$

which can be written as

$$k\sqrt{n}j_p'(k\sqrt{n})j_p(k) - j_p(k\sqrt{n})(kj_p'(k) + \eta j_p(k)) = 0 \quad (16)$$

where  $j_p$  denotes the spherical Bessel function of the first kind of order  $p$ .

Now notice that the direct scattering problem (1)–(2) can be written as find  $u \in H^1(D)$  and  $u^s \in H_{loc}^1(\mathbb{R}^3 \setminus \overline{D})$  such that

$$\begin{aligned} \Delta u + k^2 n u &= 0 \text{ in } D \quad \text{and} \quad \Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \\ u^s - u &= -e^{ikx \cdot d} \quad \text{and} \quad \frac{\partial u^s}{\partial r} + \eta u^s - \frac{\partial u}{\partial r} = -\left(\frac{\partial}{\partial r} + \eta\right) e^{ikx \cdot d} \quad \text{on } \partial D \end{aligned}$$

where  $u$  is the total field in  $D$  and  $u^s$  is the radiating scattered field in  $\mathbb{R}^3 \setminus \overline{D}$ . We make the ansatz that the solutions can be written as the following series

$$u^s(x) = \sum_{p=0}^{\infty} \sum_{m=-p}^p \alpha_p^m h_p^{(1)}(k|x|) Y_p^m(\hat{x}) \quad \text{and} \quad u(x) = \sum_{p=0}^{\infty} \sum_{m=-p}^p \beta_p^m j_p(k\sqrt{n}|x|) Y_p^m(\hat{x})$$

where  $\hat{x} = x/|x|$ . Recall the Jacobi-Anger series expansion for plane waves given by

$$e^{ikx \cdot d} = 4\pi \sum_{p=0}^{\infty} \sum_{m=-p}^p i^p j_p(k|x|) Y_p^m(\hat{x}) \overline{Y_p^m(d)}.$$

Here we let  $h_p^{(1)}$  denote the spherical Hankel function of the first kind of order  $p$  and  $Y_p^m$  is the spherical harmonic. Applying the boundary condition on  $\partial D$  gives the  $2 \times 2$  system

$$\begin{bmatrix} h_p^{(1)}(k) & -j_p(k\sqrt{n}) \\ kh_p^{(1)'}(k) + \eta h_p^{(1)}(k) & -k\sqrt{n}j_p'(k\sqrt{n}) \end{bmatrix} \begin{bmatrix} \alpha_p^m \\ \beta_p^m \end{bmatrix} = -4\pi i^p \overline{Y_p^m(d)} \begin{bmatrix} j_p(k) \\ kj_p'(k) + \eta j_p(k) \end{bmatrix}.$$

---

Solving the system gives that  $\alpha_p^m = -4\pi i^p \overline{Y_p^m(d)} \lambda_p$  where

$$\lambda_p = \frac{k\sqrt{n}j_p'(k\sqrt{n})j_p(k) - j_p(k\sqrt{n})(kj_p'(k) + \eta j_p(k))}{k\sqrt{n}j_p'(k\sqrt{n})h_p^{(1)}(k) - j_p(k\sqrt{n})(kh_p^{(1)'}(k) + \eta h_p^{(1)}(k))}.$$

Therefore the far field pattern is given by (see Theorem 2.16 of [9])

$$u^\infty(\hat{x}, d) = \frac{1}{k} \sum_{p=0}^{\infty} \sum_{m=-p}^p \frac{1}{i^{p+1}} \alpha_p^m Y_p^m(\hat{x}) = \frac{4\pi i}{k} \sum_{p=0}^{\infty} \sum_{m=-p}^p \lambda_p Y_p^m(\hat{x}) \overline{Y_p^m(d)}.$$

Using the series expansion of the far field pattern we have that by interchanging summation and integration

$$(\mathcal{F}g_z)(\hat{x}) = \int_{\mathbb{S}} u^\infty(\hat{x}, d) g(d) \, ds(d) = \frac{4\pi i}{k} \sum_{p=0}^{\infty} \sum_{m=-p}^p \lambda_p g_p^m Y_p^m(\hat{x})$$

where we define  $g_p^m = (g, Y_p^m(d))_{L^2(\mathbb{S})}$  the Fourier coefficients of  $g$ . This implies that the solution to the far field equation is given by

$$g_z(d) = \frac{k}{i} \sum_{p=0}^{\infty} \sum_{m=-p}^p i^p \frac{j_p(k|z|)}{\lambda_p} \overline{Y_p^m(\hat{z})} Y_p^m(d).$$

Notice that  $\lambda_p \rightarrow 0$  as  $k \rightarrow k_p$  where  $k_p$  is an interior conductive eigenvalue corresponding to a solution of (16). This implies that at least one of the Fourier coefficients  $g_p^m$  becomes unbounded which gives that  $\|g\|_{L^2(\mathbb{S})}^2 \rightarrow \infty$  as  $k$  approaches an interior conductive eigenvalue.

## 5 Convergence of the interior conductive eigenvalues

In this section, we consider the limiting case as  $\|\eta(x)\|_{L^\infty(\partial D)} \rightarrow 0$  for the conductive eigenvalue problem (4)–(5). We will prove that the real interior conductive eigenvalues  $k_\eta$  converge to the interior transmission eigenvalues (where  $\eta = 0$ ). We will also consider the convergence of the eigenfunctions. In our analysis, we will focus on the case where  $n_{\min} > 1$  (the analysis for  $0 < n_{\max} < 1$  follows from similar arguments).

We consider the conductive eigenvalue problem (6)–(7) for a nontrivial function in  $H^2(D) \cap H_0^1(D)$ . Now define the following bounded sesquilinear forms  $\mathcal{A}_{\eta,k}(\cdot, \cdot)$  and  $\mathcal{B}(\cdot, \cdot)$  on  $H^2(D) \cap H_0^1(D) \times H^2(D) \cap H_0^1(D)$  by

$$\mathcal{A}_{\eta,k}(u, \varphi) = \int_D \frac{1}{n-1} (\Delta u + k^2 u) (\Delta \bar{\varphi} + k^2 \bar{\varphi}) + k^4 u \bar{\varphi} \, dx + \int_{\partial D} \frac{k^2}{\eta} \frac{\partial u}{\partial \nu} \frac{\partial \bar{\varphi}}{\partial \nu} \, ds \quad (17)$$

---

and

$$\mathcal{B}(u, \varphi) = \int_D \nabla u \cdot \nabla \bar{\varphi} \, dx. \quad (18)$$

Therefore, we have that the pair  $(k_\eta, u_\eta) \in \mathbb{R}^+ \times H^2(D) \cap H_0^1(D)$  is an eigenpair provided that

$$\mathcal{A}_{\eta, k_\eta}(u_\eta, \varphi) - k_\eta^2 \mathcal{B}(u_\eta, \varphi) = 0 \quad \text{for all } \varphi \in H^2(D) \cap H_0^1(D). \quad (19)$$

For the case where  $\eta = 0$  we define

$$\mathcal{A}_{0, k}(u, \varphi) = \int_D \frac{1}{n-1} (\Delta u + k^2 u) (\Delta \bar{\varphi} + k^2 \bar{\varphi}) + k^4 u \bar{\varphi} \, dx \quad (20)$$

and the pair  $(k_0, u_0) \in \mathbb{R}^+ \times H_0^2(D)$  is an eigenpair for  $\eta = 0$  provided that

$$\mathcal{A}_{0, k_0}(u_0, \varphi) - k_0^2 \mathcal{B}(u_0, \varphi) = 0 \quad \text{for all } \varphi \in H_0^2(D) \quad (21)$$

where we define the Sobolev space

$$H_0^2(D) = \left\{ u \in H^2(D) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}$$

equipped with the  $H^2(D)$  norm.

The real interior conductive eigenvalues satisfy

$$\lambda_j(k, \eta) - k^2 = 0 \quad \text{for all } \eta \geq 0 \quad (22)$$

where

$$\lambda_j(k, \eta) = \min_{U \in \mathcal{U}_j} \max_{u \in U \setminus \{0\}} \frac{\mathcal{A}_{\eta, k}(u, u)}{\mathcal{B}(u, u)} \quad \text{for all } \eta \geq 0. \quad (23)$$

Here  $\mathcal{U}_j$  is the set of all  $j$ -dimensional subspaces of  $H^2(D) \cap H_0^1(D)$ . By Theorem 5.1 and Corollary 5.3 in [1] we have that the interior conductive eigenvalues  $k_\eta$  are an increasing sequence of  $\eta$  provided that  $\eta$  is decreasing. We now show that the sequence of interior conductive eigenvalues are bounded with respect to  $\eta > 0$ .

**Theorem 5.1.** *There exists an infinite sequence of interior conductive eigenvalues  $k_\eta^{(j)}$  for  $\eta > 0$  with  $j \in \mathbb{N}$  satisfying  $0 < k_\eta^{(j)} \leq k_0^{(j)}$  where  $k_0^{(j)}$  is an interior transmission eigenvalue for  $\eta = 0$ .*



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*Proof.* Just as in Theorem 5.1 in [1] we have that the Rayleigh quotient (23) implies that  $\lambda_j(k, \eta) \leq \lambda_j(k, 0)$  for all  $\eta > 0$ . Therefore, we can conclude that

$$\lambda_j(k_0, \eta) - k_0^2 \leq \lambda_j(k_0, 0) - k_0^2 = 0$$

where  $k_0$  is an interior transmission eigenvalue for  $\eta = 0$ . Recall that the real interior transmission eigenvalues satisfy (22) and that the mapping

$$k \longmapsto \lambda_j(k, \eta) - k^2 \quad \text{for all } \eta \geq 0$$

is continuous for  $k \in (0, \infty)$ . By Theorem 4.3 in [1] we have that there exists a  $\delta$  (independent of  $\eta$ ) such that for all  $k > \delta$

$$\lambda_j(k, \eta) - k^2 > 0$$

and therefore we can conclude that at least one root of (22) is in the interval  $(\delta, k_0]$  for every  $j \in \mathbb{N}$ , proving the claim.  $\square$

We now have that the sequence of interior conductive eigenvalues  $\{k_\eta\}_{\eta>0}$  is a bounded monotonic sequence and is therefore convergent (up to a subsequence if  $\eta$  is not decreasing). Now let  $k \in \mathbb{R}^+$  be such that  $k_\eta \rightarrow k$  as  $\eta \rightarrow 0$ . Since the corresponding eigenfunctions  $u_\eta$  are nontrivial we assume that they are normalized in the  $H^1(D)$  norm. In [1] it is shown that  $\mathcal{A}_{\eta,k}(\cdot, \cdot)$  is coercive on  $H^2(D) \cap H_0^1(D)$  where the coercivity constant is independent of  $k$  and  $\eta$ . Therefore, we have that there is a constant  $\alpha > 0$  where

$$\alpha \|u_\eta\|_{H^2(D)}^2 \leq \mathcal{A}_{\eta,k_\eta}(u_\eta, u_\eta) = k_\eta^2 \mathcal{B}(u_\eta, u_\eta) = k_\eta^2 \|\nabla u_\eta\|_{L^2(D)}^2 \leq k_0^2$$

which gives that  $u_\eta$  is a bounded sequence in  $H^2(D)$  and is therefore weakly convergent to some  $u \in H^2(D) \cap H_0^1(D)$  (strongly in  $H^1(D)$ ). Notice, that since  $\|u_\eta\|_{H^1(D)} = 1$  for all  $\eta > 0$  the strong  $H^1(D)$  convergence implies that the limit has unit  $H^1(D)$  norm and therefore  $u \neq 0$ .

We now want to show that the limiting pair  $(k, u)$  is a transmission eigenpair for  $\eta = 0$ . Notice that since  $u_\eta \in H^2(D) \cap H_0^1(D)$  the compact embedding of  $H^{1/2}(\partial D)$  into  $L^2(\partial D)$  implies that  $\partial_\nu u_\eta$  converges strongly to  $\partial_\nu u$  in  $L^2(\partial D)$ . By equation (19) we obtain that

$$k_\eta^2 \int_{\partial D} \frac{1}{\eta} \left| \frac{\partial u_\eta}{\partial \nu} \right|^2 ds = k_\eta^2 \mathcal{B}(u_\eta, u_\eta) - \mathcal{A}_{0,k_\eta}(u_\eta, u_\eta).$$

Using the continuity of the sesquilinear forms and the fact that both  $k_\eta$  and  $u_\eta$  are bounded sequences we conclude that

$$0 \leq k_\eta^2 \left\| \frac{\partial u_\eta}{\partial \nu} \right\|_{L^2(\partial D)}^2 \leq C \eta_{max} \quad \text{where} \quad \sup_{\partial D} \eta(x) = \eta_{max} > 0.$$

Notice that the constant  $C$  depends only on  $n$ ,  $k_0$  and  $\alpha$ . Notice, that the above inequality gives that  $\partial_\nu u_\eta$  converges strongly to zero in  $L^2(\partial D)$  which implies that  $u \in H_0^2(D)$ . Therefore, by the convergence of the eigenvalues  $k_\eta$  and weak convergence of the eigenfunctions  $u_\eta$  if we take any  $\varphi \in H_0^2(D)$

$$\begin{aligned} 0 &= \lim_{\eta \rightarrow 0} \mathcal{A}_{\eta, k_\eta}(u_\eta, \varphi) - k_\eta^2 \mathcal{B}(u_\eta, \varphi) \\ &= \lim_{\eta \rightarrow 0} \mathcal{A}_{0, k_\eta}(u_\eta, \varphi) - k_\eta^2 \mathcal{B}(u_\eta, \varphi) \\ &= \mathcal{A}_{0, k}(u, \varphi) - k^2 \mathcal{B}(u, \varphi). \end{aligned}$$

By the above analysis we have the following result.

**Theorem 5.2.** *Let  $(k_\eta, u_\eta) \in \mathbb{R}^+ \times H^2(D) \cap H_0^1(D)$  be an interior conductive eigenpair for  $\eta > 0$ . If  $k_\eta$  is bounded with respect to  $\eta$  then as  $\|\eta(x)\|_{L^\infty(\partial D)} \rightarrow 0$  there is a subsequence of  $k_\eta$  and  $u_\eta$  such that  $k_\eta \rightarrow k_0$  and  $u_\eta$  converges weakly to  $u_0$  in  $H^2(D)$  where  $(k_0, u_0) \in \mathbb{R}^+ \times H_0^2(D)$  is an interior transmission eigenpair for  $\eta = 0$ .*

Theorem 5.2 verifies the conjecture in [1] that the interior conductive eigenvalues converge as  $\eta \rightarrow 0$  to the interior transmission eigenvalues. In [1] there are some numerical experiments that indicate that the order of convergence is one.

We will now show that the eigenfunctions converge strongly with respect to the  $H^2(D)$  norm. To prove the strong convergence we will use the coercivity of the sesquilinear form  $\mathcal{A}_{\eta, k}(\cdot, \cdot)$ . To this end, let  $(k_\eta, u_\eta) \in \mathbb{R}^+ \times H^2(D) \cap H_0^1(D)$  be an interior conductive eigenpair for  $\eta > 0$  and  $(k_0, u_0) \in \mathbb{R}^+ \times H_0^2(D)$  is the limit corresponding to an interior transmission eigenpair for  $\eta = 0$ . Now for  $\varphi \in H^2(D) \cap H_0^1(D)$  we have that

$$\begin{aligned} \mathcal{A}_{\eta, k_\eta}(u_\eta - u_0, \varphi) &= \mathcal{A}_{\eta, k_\eta}(u_\eta, \varphi) - \mathcal{A}_{0, k_\eta}(u_0, \varphi) \\ &= k_\eta^2 \mathcal{B}(u_\eta, \varphi) - k_0^2 \mathcal{B}(u_0, \varphi) + (\mathcal{A}_{0, k_0} - \mathcal{A}_{0, k_\eta})(u_0, \varphi) \\ &= (k_\eta^2 - k_0^2) \mathcal{B}(u_\eta, \varphi) + k_0^2 \mathcal{B}(u_\eta - u_0, \varphi) + (\mathcal{A}_{0, k_0} - \mathcal{A}_{0, k_\eta})(u_0, \varphi) \end{aligned}$$

where we have used the variational formulation of the conductive eigenvalue problems (19) and (21) along with the fact that  $u_0$  is in  $H_0^2(D)$ . We now estimate the terms on the right hand side where  $\varphi = u_\eta - u_0$ . Therefore, simple calculations gives that

$$\begin{aligned} &(\mathcal{A}_{0, k_0} - \mathcal{A}_{0, k_\eta})(u_0, \varphi) \\ &= (k_0^2 - k_\eta^2) \int_D \frac{1}{n-1} (\bar{\varphi} \Delta u_0 + u_0 \Delta \bar{\varphi}) dx + (k_0^4 - k_\eta^4) \int_D \frac{n}{n-1} u_0 \bar{\varphi} dx. \end{aligned}$$

Now, letting  $\varphi = u_\eta - u_0$  and using the fact that  $u_0$  and  $u_\eta$  are bounded with respect to  $\eta$  gives that

$$\left| (\mathcal{A}_{0, k_0} - \mathcal{A}_{0, k_\eta})(u_0, u_\eta - u_0) \right| \leq C_1 |k_0^2 - k_\eta^2| + C_2 |k_0^4 - k_\eta^4| \longrightarrow 0 \quad \text{as } \eta \rightarrow 0$$

where the positive constants  $C_1$  and  $C_2$  are independent of  $\eta$ . Similarly we can conclude that

$$\left| (k_\eta^2 - k_0^2) \mathcal{B}(u_\eta, u_\eta - u_0) \right| \leq C_3 |k_0^2 - k_\eta^2| \longrightarrow 0 \quad \text{as } \eta \rightarrow 0$$

where the positive constant  $C_3$  is independent of  $\eta$ . Using the fact that  $u_\eta - u_0$  converges strongly to zero in  $H^1(D)$  gives that

$$k_0^2 \mathcal{B}(u_\eta - u_0, u_\eta - u_0) = k_0^2 \|\nabla(u_\eta - u_0)\|_{L^2(D)}^2 \longrightarrow 0 \quad \text{as } \eta \rightarrow 0.$$

By using the coercivity of  $\mathcal{A}_{\eta,k}(\cdot, \cdot)$  we obtain that

$$\begin{aligned} \alpha \|u_\eta - u_0\|_{H^2(D)}^2 &\leq \mathcal{A}_{\eta,k_\eta}(u_\eta - u_0, u_\eta - u_0) \\ &= (\mathcal{A}_{0,k_0} - \mathcal{A}_{0,k_\eta})(u_0, u_\eta - u_0) \\ &\quad + (k_\eta^2 - k_0^2) \mathcal{B}(u_\eta, u_\eta - u_0) + k_0^2 \mathcal{B}(u_\eta - u_0, u_\eta - u_0) \end{aligned}$$

giving that  $u_\eta - u_0$  converges strongly to zero in  $H^2(D)$ .

**Theorem 5.3.** *Let  $(k_\eta, u_\eta) \in \mathbb{R}^+ \times H^2(D) \cap H_0^1(D)$  be an interior conductive eigenpair for  $\eta > 0$ . If  $k_\eta$  is bounded with respect to  $\eta$  then as  $\|\eta(x)\|_{L^\infty(\partial D)} \rightarrow 0$  there is a subsequence of  $k_\eta$  and  $u_\eta$  such that  $k_\eta \rightarrow k_0$  and  $u_\eta$  converges strongly to  $u_0$  in  $H^2(D)$  where  $(k_0, u_0) \in \mathbb{R}^+ \times H_0^2(D)$  is an interior transmission eigenpair for  $\eta = 0$ .*

## 5.1 Convergence rate for radially symmetric eigenfunctions

We now look at the radially symmetric eigenfunctions and prove the linear convergence rate as predicted by our numerical examples (see Section 6). Notice that

$$w_\eta(r) = J_0(k_\eta) J_0(k_\eta \sqrt{n}r) \quad \text{and} \quad v_\eta(r) = J_0(k_\eta \sqrt{n}) J_0(k_\eta r)$$

are the eigenfunctions for the unit circle with  $k_\eta \in \mathbb{R}$  the corresponding eigenvalues. We will prove the linear convergence rate for  $w_\eta$  and the analysis is similar for the other eigenfunction. To this end, we assume that the eigenvalues have a linear convergence rate and we therefore have

$$\begin{aligned} |w_\eta(r) - w_0(r)| &= |J_0(k_\eta) J_0(k_\eta \sqrt{n}r) - J_0(k_0) J_0(k_0 \sqrt{n}r)| \\ &\leq |J_0(k_\eta) J_0(k_\eta \sqrt{n}r) - J_0(k_0) J_0(k_\eta \sqrt{n}r)| + |J_0(k_0) J_0(k_\eta \sqrt{n}r) - J_0(k_0) J_0(k_0 \sqrt{n}r)|, \end{aligned}$$

where  $k$  is a transmission eigenvalue for  $\eta = 0$ . Since  $J_0$  is bounded by one and the Mean Value Theorem

$$\begin{aligned} |w_\eta(r) - w_0(r)| &\leq |J_0(k_\eta) - J_0(k_0)| + |J_0(k_\eta \sqrt{n}r) - J_0(k_0 \sqrt{n}r)| \\ &\leq |J_1(\xi_1)| |k_\eta - k_0| + \sqrt{n} r |J_1(\xi_2)| |k_\eta - k_0|, \end{aligned}$$

---

where  $k_\eta < \xi_1 < k_0$  and  $k_\eta\sqrt{n}r < \xi_2 < k_0\sqrt{n}r$  with  $0 \leq r \leq 1$ . Similarly, since  $J_1$  is bounded by one, we have that

$$|w_\eta(r) - w_0(r)| \leq C|k_\eta - k_0|$$

where  $C > 0$  is a positive constant that is independent of  $\eta$ . Now, since we have that the eigenvalues have a linear convergence rate, we can conclude that the eigenfunctions also have a linear convergence rate. Also note that the case for spherical eigenfunctions in  $\mathbb{R}^3$  works similarly.

We now consider the case of complex eigenvalues so we assume that  $k_\eta \in \mathbb{C}$ . Before we begin, we need a generalization of the Mean Value Theorem for holomorphic functions. To this end, assume that  $f(z)$  is a holomorphic function and let  $a$  and  $b$  be distinct points. Now define the real valued functions

$$F(t) = \operatorname{Re}\left(\overline{(b-a)}f(a+t(b-a))\right) \quad \text{and} \quad G(t) = \operatorname{Im}\left(\overline{(b-a)}f(a+t(b-a))\right)$$

for  $0 \leq t \leq 1$ . Applying the Mean Value Theorem to the functions  $F$  and  $G$  implies that there exists  $\alpha, \beta$  on the line segment connecting  $a$  and  $b$  such that

$$|b-a|^2 \operatorname{Re}(f'(\alpha)) = \operatorname{Re}\left(\overline{(b-a)}(f(b) - f(a))\right)$$

and

$$|b-a|^2 \operatorname{Im}(f'(\beta)) = \operatorname{Im}\left(\overline{(b-a)}(f(b) - f(a))\right).$$

Now recall that

$$\begin{aligned} w_\eta(r) - w_0(r) &= J_0(k_\eta)J_0(k_\eta\sqrt{n}r) - J_0(k_0)J_0(k_0\sqrt{n}r) \\ &= J_0(k_\eta\sqrt{n}r)\left(J_0(k_\eta) - J_0(k_0)\right) + J_0(k_0)\left(J_0(k_\eta\sqrt{n}r) - J_0(k_0\sqrt{n}r)\right). \end{aligned}$$

Therefore, by the generalization of the Mean Value Theorem

$$|k_\eta - k_0|^2 \operatorname{Re}(J_1(\alpha)) = \operatorname{Re}\left(\overline{(k_0 - k_\eta)}(J_0(k_\eta) - J_0(k_0))\right) \quad (24)$$

and

$$|k_\eta - k_0|^2 \operatorname{Im}(J_1(\beta)) = \operatorname{Im}\left(\overline{(k_0 - k_\eta)}(J_0(k_\eta) - J_0(k_0))\right) \quad (25)$$

where  $\alpha, \beta$  are on the line segment connecting  $k_\eta$  and  $k_0$ . Since  $J_0$  is analytic this implies that the right hand sides of equations (24) and (25) are of order  $|k_\eta - k_0|^2 = \mathcal{O}(\eta^2)$  and therefore so is the left hand side which gives that  $|J_0(k_\eta) - J_0(k_0)| = \mathcal{O}(\eta)$ . Similarly one can show that  $|J_0(k_\eta\sqrt{n}r) - J_0(k_0\sqrt{n}r)| = \mathcal{O}(\eta)$ . Therefore, we have that

$$|w_\eta(r) - w_0(r)| = \mathcal{O}(\eta)$$

---

for any  $0 \leq r \leq 1$  proving the linear convergence that is seen in Section 6. This analysis also works in  $\mathbb{R}^3$ .

The main assumption in this section is that the interior conductive eigenvalues converge linearly to the interior transmission eigenvalues as  $\eta \rightarrow 0$ . It has been proven that the real interior conductive eigenvalues converge to the real interior transmission eigenvalues but the convergence rate is still an open question. We now give some analytical evidence that the convergence rate is linear for the eigenvalues corresponding to the radially symmetric eigenfunctions. Recall, that for the unit sphere in  $\mathbb{R}^3$  that the eigenvalues are the roots of

$$d_\eta(k) = k \sin(k\sqrt{n}) \cos(k) - k\sqrt{n} \sin(k) \cos(k\sqrt{n}) + \eta \sin(k) \sin(k\sqrt{n}).$$

Therefore, we define that function

$$f(k) = k \cot(k) - k\sqrt{n} \cot(k\sqrt{n})$$

where it is clear that  $k_\eta$  is an interior conductive eigenvalue if  $f(k_\eta) = \eta$  and  $k_0$  is an interior transmission eigenvalue provided that  $f(k_0) = 0$ . Now, assume that the transmission eigenvalue  $k_0$  is contained in an interval where  $f$  is a  $C^1$  function and that  $f'(k_0) \neq 0$ , by the Inverse Function Theorem there is an open set containing  $k_0$  where  $f$  has a  $C^1$  inverse. We denote the inverse of  $f$  by  $g$  which implies that  $k_\eta = g(\eta)$ . Therefore, appealing to the Mean Value Theorem we can conclude that  $|k_\eta - k_0| = |g'(\xi)|\eta$  for all  $\eta$  sufficiently small where  $k_\eta < \xi < k_0$ . Since  $g$  is  $C^1$  in an interval we can conclude that  $|k_\eta - k_0| = \mathcal{O}(\eta)$  for all  $\eta > 0$  sufficiently small.

## 6 Numerical results

### 6.1 Complex-valued interior conductive eigenvalues

We have shown in Section 3 that under certain conditions we are able to find infinitely many complex-valued interior conductive eigenvalues for a unit sphere and a unit circle provided that  $n$  and  $\eta$  are chosen as constants. We choose  $n = 3$  and plot the real and complex-valued ICEs in the domain  $\Omega_1 = [1, 10] \times [-10, 10]i \subset \mathbb{C}$  for the unit sphere and the unit circle using  $\eta = 1$ ,  $\eta = 1/2$ , and  $\eta = 0$  as illustrated in Fig. 1. As we can see, we are able to locate two real and three complex-valued ICEs (including their complex-conjugate counterpart) for both the unit sphere and the unit circle. Next, we choose  $n = 1/3$ , the same parameters  $\eta$  and the domain  $\Omega_2 = [0, 10] \times [-10, 10]i \subset \mathbb{C}$ . The results are shown in Fig. 2. We are able to locate two real and one complex-valued ICE (including their complex-conjugate counterpart) for the unit sphere. For the unit circle we have one more complex-value ICE. Note that there is no purely imaginary ICE as proven in Theorem 3.4. Finally, we plot as one example also the eigenfunction  $w$  and  $v$  for a unit sphere using  $n = 3$  and  $\eta = 1$  with  $k = 2.822\,203 + 0.720\,745i$ . We only plot a quarter of a slice in the  $xy$ -plane as illustrated in Figure 3, since the eigenfunctions are radially symmetric.

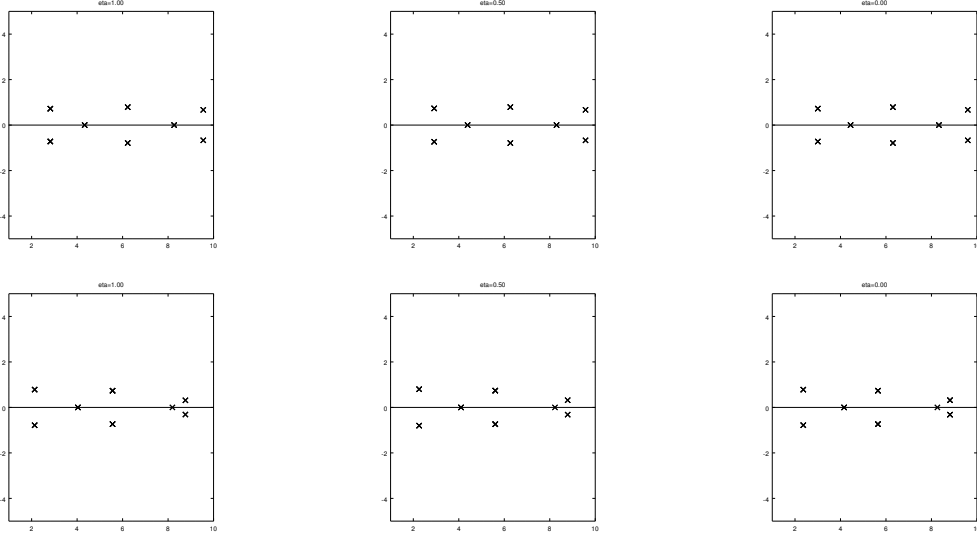


Figure 1: First row: ICEs for a unit sphere. Second row: ICEs for a unit circle. Parameters are  $n = 3$  and  $\eta = 1$ ,  $\eta = 1/2$ , and  $\eta = 0$ .

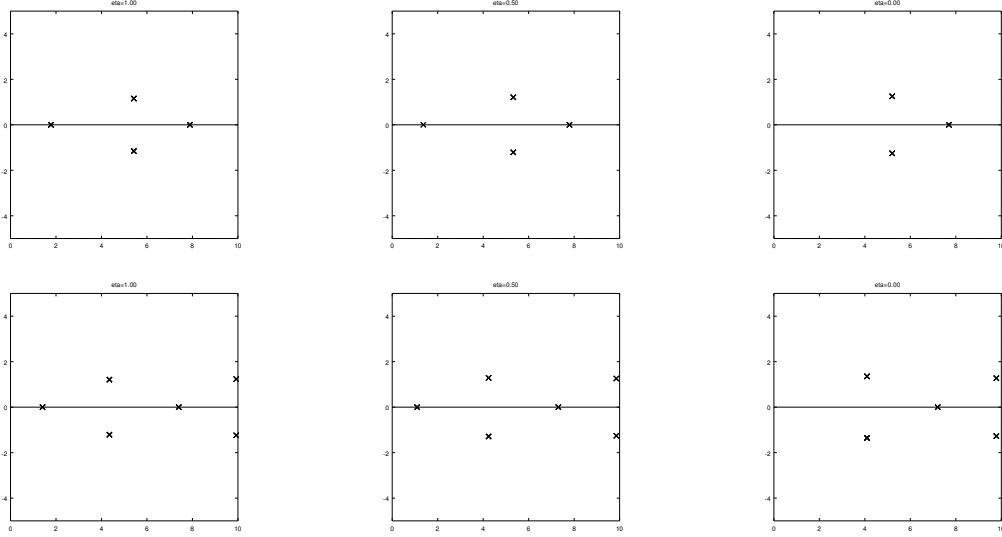


Figure 2: First row: ICEs for a unit sphere. Second row: ICEs for a unit circle. Parameters are  $n = 1/3$  and  $\eta = 1$ ,  $\eta = 1/2$ , and  $\eta = 0$ .

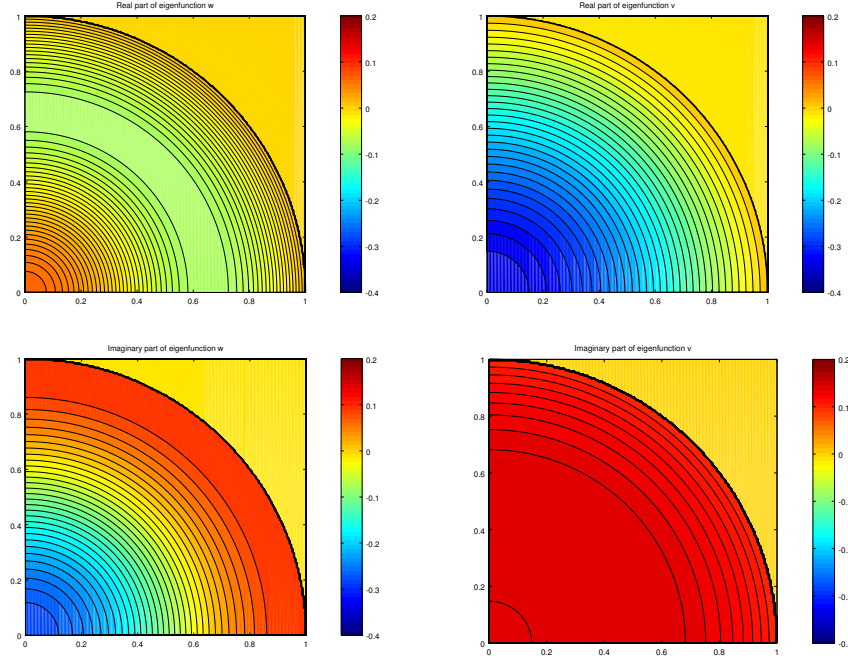


Figure 3: First row: Real part of  $w$  and  $v$  for a unit sphere. Second row: Imaginary part of  $w$  and  $v$  for a unit circle. Parameters are  $n = 3$  and  $\eta = 1$ .

## 6.2 Convergence of the interior conductive eigenvalues and eigenfunctions

As we have proven in Section 5, the interior conductive eigenvalues (ICEs)  $k_\eta$  converge to the interior transmission eigenvalues (ITEs)  $k_0$  as  $\eta$  approach 0. This is true for both real and complex-valued ICEs. For the index of refraction  $n = 3$  and the sequence  $\eta = 1/2^i$  with  $i = 0, 1, \dots, 8$ , we obtain two real and three complex-valued ICEs in the domain  $\Omega = [0, 10] \times [0, 10]i \subset \mathbb{C}$  using a unit sphere. The ITEs are 4.443 358, 8.328 578,  $3.003\,079 + 0.723\,476i$ ,  $6.305\,573 + 0.787\,309i$ ,  $9.598\,536 + 0.669\,770i$ . With the absolute error  $\epsilon_\eta^{(j)} = |k_0^{(j)} - k_\eta^{(j)}|$  for the  $j$ -th ICE we define the estimated order of convergence for the  $j$ -th ICE by  $\text{EOC}^{(j)} = \log(\epsilon_\eta^{(j)} / \epsilon_{\eta/2}^{(j)}) / \log(2)$ .

In Table 1, we show the estimated order of convergence for the five ICEs.

Interestingly, the order of convergence seems to be linear for both the real and complex-valued ICEs. Note that the conjecture of the linear convergence for constant  $n$  has already been made in [1, Table 3] for the real-valued ICEs, but the proof was still open. Next, we show numerically that the eigenfunctions  $w_\eta$  and  $v_\eta$  for the ICEs converge to the eigenfunctions  $w_0$  and  $v_0$  of the interior transmission eigenvalues, respectively. Therefore, we need

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$\eta$	EOC <sup>(1)</sup>	EOC <sup>(2)</sup>	EOC <sup>(3)</sup>	EOC <sup>(4)</sup>	EOC <sup>(5)</sup>
1					
1/2	0.977	1.023	1.068	1.007	0.991
1/4	0.993	1.013	1.044	1.006	0.997
1/8	0.998	1.007	1.024	1.004	0.999
1/16	0.999	1.003	1.012	1.002	0.999
1/32	1.000	1.002	1.006	1.001	1.000
1/64	1.000	1.001	1.003	1.001	1.000
1/128	1.000	1.000	1.002	1.000	1.000
1/256	1.000	1.000	1.001	1.000	1.000

Table 1: The estimated order of convergence for five interior conductive eigenvalues for a unit sphere using  $n = 3$  as  $\eta \rightarrow 0$ .

the squared  $L^2$  error for the unit sphere which is given as

$$\epsilon_\eta^{(w)} = 4\pi \int_0^1 |w_\eta(r) - w_0(r)|^2 r^2 dr$$

with the obvious definition of  $\epsilon_\eta^{(v)}$ . Note that we used  $c_1 = j_0(k_\eta)$  and  $c_2 = j_0(k_\eta\sqrt{n})$  in the definition of the two eigenfunctions. The estimated order of convergence is given by

$$\text{EOC}^{(w)} = \log \left( \sqrt{\epsilon_\eta^{(w)} / \epsilon_{\eta/2}^{(w)}} \right) / \log(2) = \log (\epsilon_\eta^{(w)} / \epsilon_{\eta/2}^{(w)}) / (2 \log(2))$$

and likewise  $\text{EOC}^{(v)}$ . In Table 2 we show the estimated order of convergence for the first real and the first complex-valued ICEs. As we can see the estimated order of convergence

$\eta$	EOC <sup>(w)</sup>	EOC <sup>(v)</sup>	EOC <sup>(w)</sup>	EOC <sup>(v)</sup>
1				
1/2	0.992	1.004	1.093	1.075
1/4	1.002	1.000	1.071	1.063
1/8	1.002	0.999	1.040	1.036
1/16	1.001	1.000	1.021	1.019
1/32	1.001	1.000	1.010	1.010
1/64	1.000	1.000	1.005	1.005
1/128	1.000	1.000	1.003	1.002
1/256	1.000	1.000	1.001	1.001

Table 2: The estimated order of convergence for two pairs of conductive eigenfunctions for a unit sphere using  $n = 3$  as  $\eta \rightarrow 0$ .



is linear. The proof was still open.

Next, we use a unit circle with the same parameters as before. Again, we obtain two real and three complex-valued ICEs in the domain  $\Omega = [0, 10] \times [0, 10]i \subset \mathbb{C}$ . The ITEs are 4.159 236, 8.261 173, 2.363 421+0.781 661i, 5.646 922+0.735 262i, and 8.814 961+0.318 519i. As we can see in Table 3 the estimated order of convergence for the five ICES is linear.

$\eta$	EOC <sup>(1)</sup>	EOC <sup>(2)</sup>	EOC <sup>(3)</sup>	EOC <sup>(4)</sup>	EOC <sup>(5)</sup>
1					
1/2	1.018	1.108	1.081	0.997	0.950
1/4	1.014	1.055	1.056	1.002	0.977
1/8	1.009	1.028	1.030	1.002	0.990
1/16	1.005	1.014	1.015	1.001	0.996
1/32	1.002	1.007	1.008	1.001	0.997
1/64	1.001	1.003	1.004	1.000	1.007
1/128	1.001	1.002	1.002	0.999	0.989
1/256	1.000	1.001	1.001	1.000	0.999

Table 3: The estimated order of convergence for five interior conductive eigenvalues for a unit circle using  $n = 3$  as  $\eta \rightarrow 0$ .

For the unit circle, we define the squared  $L^2$  error by

$$\epsilon_{\eta}^{(w)} = 2\pi \int_0^1 |w_{\eta}(r) - w_0(r)|^2 r \, dr$$

with the obvious definition of  $\epsilon_{\eta}^{(v)}$  and the appropriate use of the Bessel function  $J_0$  instead of  $j_0$ . In Table 4 we show the estimated order of convergence for the first real and the first complex-valued ICEs. We again observe a linear convergence rate.

### 6.3 Interior conductive eigenvalues for an absorbing media

In this section, we report interior conductive eigenvalues for a unit sphere and a unit circle using  $n = n_1 + in_2/k$  with  $n_1 = 3$  and  $n_2 = 1$  using  $\eta = 0$  and  $\eta = 1$  in the domain  $\Omega = [0, 10] \times [0, 10]i \subset \mathbb{C}$ . We obtain for the unit sphere the three complex-valued ICEs 3.091 338+0.669 795i, 6.318 574+0.736 283i, and 9.541 381+0.648 605i for  $\eta = 0$ . For  $\eta = 1$ , we get 2.925 606+0.685 981i, 6.239 500+0.741 293i, and 9.489 189+0.650 159i. The results for the unit circle are 2.420 303+0.713 169i, 5.625 863+0.695 149i, and 8.696 325+0.525 1402i for  $\eta = 0$  and 2.209 319+0.744 904i, 5.537 406+0.700 587i, and 8.638 889+0.525 763i for  $\eta = 1$ , respectively.

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$\eta$	EOC <sup>(w)</sup>	EOC <sup>(v)</sup>	EOC <sup>(w)</sup>	EOC <sup>(v)</sup>
1				
1/2	1.034	0.941	1.082	1.071
1/4	1.022	0.967	1.083	1.078
1/8	1.012	0.983	1.047	1.046
1/16	1.006	0.992	1.025	1.024
1/32	1.003	0.996	1.013	1.012
1/64	1.002	0.998	1.006	1.006
1/128	1.001	0.999	1.003	1.003
1/256	1.000	0.999	1.002	1.002

Table 4: The estimated order of convergence for two pairs of conductive eigenfunctions for a unit circle using  $n = 3$  as  $\eta \rightarrow 0$ .

#### 6.4 The inside-outside-duality method

Recently, a new method has been invented to determine interior transmission eigenvalues (see [18]) which has been successfully applied to various scattering problems (see [21, 24, 25, 26]). It is another approach which does not need to solve a nonlinear eigenvalue problem as done in [19, 20].

In this section, we show that we are able to determine the interior conductive eigenvalues using the inside-outside-duality approach, although the theory still has to be carried out. We refer the reader to [25] for the details of the implementation of the inside-outside-duality method. This approach connects the eigenvalues of the far field operator to the interior eigenvalues. Let  $\lambda_j(k) \in \mathbb{C}$  for  $j \in \mathbb{N}$  be the eigenvalues of  $\mathcal{F} = \mathcal{F}_k$  defined in (3) and  $\phi_j(k) = \arg\{\lambda_j(k)\}$ . In [25] it is shown that for  $\eta = 0$  the function  $\phi(k) = \max_{j \in \mathbb{N}} \phi_j(k)$  satisfies

$$\phi(k) \rightarrow \pi \quad \text{as} \quad k \rightarrow k_0 \quad \text{a transmission eigenvalue.}$$

We choose an equidistant grid  $[1, 5]$  with grid size 0.01 and plot the phase curves  $p$  against the wave number  $k$ . As we can see in Figure 4 we are able to determine five interior conductive eigenvalues for the cases  $\eta = 0.1, 0.5, 1$ , and 3.

$\eta$	1.	2.	3.	4.	5.
0.1	3.10	3.13	3.68	4.25	4.82
0.5	2.97	3.08	3.64	4.22	4.79
1	2.79	3.02	3.60	4.18	4.76
3	2.20	2.80	3.43	4.04	4.64

Table 5: The five reconstructed interior conductive eigenvalues for the unit sphere for  $n = 4$  by the inside-outside-duality method

Comparing the results with [1, Table 1] shows that we are able to obtain the interior conductive eigenvalues within the accuracy of the chosen grid size. Interestingly, the value 3.141 593 as reported in [1, Table 1] is missing. There is another important observation to be mentioned. It has been noted that the phase curves are monotonically increasing towards  $\pi$  for the interior transmission eigenvalues for various obstacles (see [25]), but for the interior conductive eigenvalues this is not the case as one can clearly see in the second row of Figure 4.

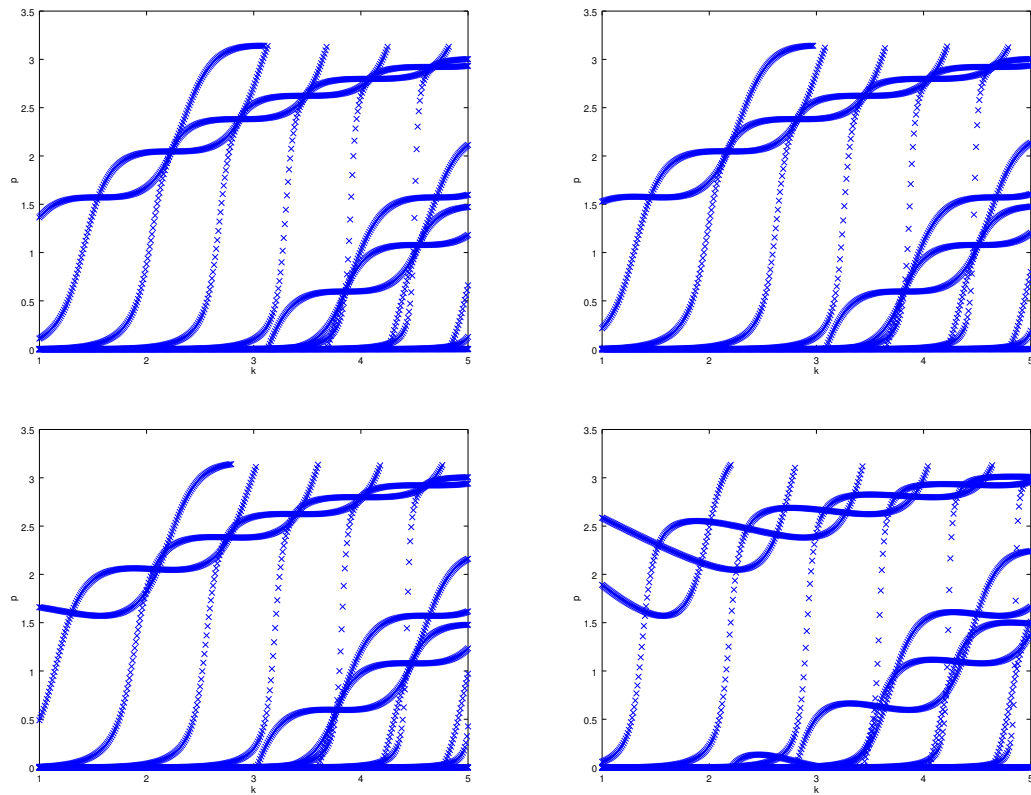


Figure 4: First row: The phase curves for a unit sphere using  $\eta = 0.1$  and  $\eta = 0.5$ . Second row: The phase curves for a unit sphere using  $\eta = 1$  and  $\eta = 3$ .

Additionally, we show that we are also able to find the interior conductive eigenvalues with the inside-outside-duality method for a unit circle using the same parameters as before. In Figure 5 we show the phase curves for the two cases  $\eta = 1$  and  $\eta = 3$ . Again, it is interesting to observe that the phase curves are several times increasing and decreasing before approaching the value  $\pi$ .

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$\eta$	1.	2.	3.	4.	5.	6.
1	2.77	3.29	3.31	3.89	4.47	—
3	2.49	3.12	3.14	3.74	4.34	4.94

Table 6: The reconstructed interior conductive eigenvalues for the unit circle for  $n = 4$  by the inside-outside-duality method

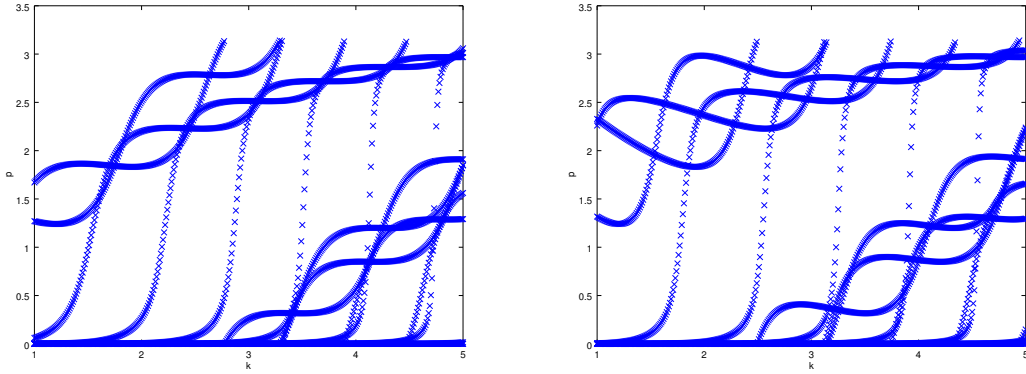


Figure 5: The phase curves for a unit circle using  $\eta = 1$  and  $\eta = 3$ .

## 7 Conclusion

In this paper, we have analyzed some of the open problems discussed in [1] for the eigenvalue problem with a conductive boundary. We began by proving the existence of complex-valued interior conductive eigenvalues for the homogenous sphere and disk. Next, we proved that the eigenvalues can be obtained by the far field data, this implies that the eigenvalues can be used for nondestructive testing. Lastly, we were able to prove that the interior conductive eigenvalue problem converges to the interior transmission eigenvalue problem as the conductivity goes to zero. Some questions that are still open for this problem are:

1. Asymptotic expansion of the eigenvalues and eigenfunctions as  $\eta \rightarrow 0$ .
2. Analysis of the interior conductive eigenvalue problem as  $\eta \rightarrow \infty$ .
3. Inverse spectral problem of reconstructing either  $n$  and/or  $\eta$  from a knowledge of the spectral data for a spherically stratified media.
4. Existence of complex-valued interior conductive eigenvalues for more general domains and coefficients.

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